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LETTER TO THE EDITOR

Exact energy distribution function in a time-dependent harmonic oscillator

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Abstract

Following a recent work by Robnik and Romanovski (2006 *J. Phys. A: Math. Gen.* **39** L35, 2006 *Open Syst. Inf. Dyn.* **13** 197–222), we derive an explicit formula for the universal distribution function of the final energies in a time-dependent 1D harmonic oscillator, whose functional form does not depend on the details of the frequency $\omega(t)$ and is closely related to the conservation of the adiabatic invariant. The normalized distribution function is $P(x) = \pi^{-1}(2\mu^2 - x^2)^{-\frac{1}{2}}$, where $x = E_1 - \bar{E}_1$; E_1 is the final energy, \bar{E}_1 is its average value and μ^2 is the variance of E_1 . \bar{E}_1 and μ^2 can be calculated exactly using the WKB approach to all orders.

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In a recent work [1, 2], Robnik and Romanovski studied the energy evolution in a general 1D time-dependent harmonic oscillator and the closely related questions of the conservation of adiabatic invariants [3–7]. Starting with the ensemble of uniformly distributed (w.r.t. the canonical angle variable) initial conditions on the initial invariant torus of energy E_0 , they calculated the average final energy \bar{E}_1 , the variance μ^2 and all the higher moments. The even moments are powers of μ^2 , whilst the odd moments are exactly zero, because the distribution function $P(E_1)$ of the final energies E_1 is an even function w.r.t. \bar{E}_1 . In this letter we derive an explicit formula for $P(E_1)$, namely we shall derive

$$P(E_1) = \operatorname{Re} \frac{1}{\pi \sqrt{2\mu^2 - x^2}}, \quad (1)$$

where $x = E_1 - \bar{E}_1$, and Re denotes the real part, so that (1) is zero for $|x| > \mu\sqrt{2}$. We do this by using the exact results for the higher (even) moments and by employing the characteristic function $f(y)$ of $P(x)$.

The dynamics of our system is described by the Newton equation

$$\ddot{q} + \omega^2(t)q = 0, \quad (2)$$

which is generated by the system's Hamilton function $H = H(q, p, t)$, whose numerical value $E(t)$ at time t is precisely the total energy of the system at time t , and in the case of a 1D harmonic oscillator this is

$$H = \frac{p^2}{2M} + \frac{1}{2}M\omega^2(t)q^2, \quad (3)$$

where q, p, M, ω are the coordinate, the momentum, the mass and the frequency of the linear oscillator, respectively.

The dynamics is linear in q, p , as described by (2), but nonlinear as a function of $\omega(t)$ and is therefore subject to the nonlinear dynamical analysis. By using the indexes 0 and 1, we denote the initial ($t = t_0$) and final ($t = t_1$) values of the variables.

The transition map Φ maps initial conditions (q_0, p_0) onto the final conditions (q_1, p_1) :

$$\Phi : \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \mapsto \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \quad (4)$$

with $\det(\Phi) = ad - bc = 1$, and a, b, c, d can be calculated as shown in [1, 2]. Let $E_0 = H(q_0, p_0, t = t_0)$ be the initial energy and $E_1 = H(q_1, p_1, t = t_1)$ be the final energy. Introducing the new coordinates, namely the action $I = E/\omega$ and the angle ϕ , and assuming the uniform distribution of initial angles ϕ over the period 2π , we can immediately calculate the final average energy \bar{E}_1 and the variance

$$\mu^2 = \overline{(E_1 - \bar{E}_1)^2} = \frac{E_0^2}{2} \left[\left(\frac{\bar{E}_1}{E_0} \right)^2 - \frac{\omega_1^2}{\omega_0^2} \right]. \quad (5)$$

It is shown in [2] that for an arbitrary positive integer m , we have $\overline{(E_1 - \bar{E}_1)^{2m-1}} = 0$ and

$$\overline{(E_1 - \bar{E}_1)^{2m}} = \frac{(2m-1)!!}{m!} \overline{(E_1 - \bar{E}_1)^2}^m. \quad (6)$$

Thus, the $2m$ th moment of $P(E_1)$ is equal to $(2m-1)!!\mu^{2m}/m!$, and therefore, indeed, all moments of $P(E_1)$ are uniquely determined by the first moment \bar{E}_1 . Obviously, $P(E_1)$ is in this sense universal because it depends only on the average final energy \bar{E}_1 and the ratio ω_1/ω_0 of the final and initial frequencies and does not depend otherwise on any details of $\omega(t)$. It has a finite support $(E_{\min}, E_{\max}) = (\bar{E}_1 - \mu\sqrt{2}, \bar{E}_1 + \mu\sqrt{2})$, it is an even distribution w.r.t. $\bar{E}_1 = (E_{\min} + E_{\max})/2$ and has an integrable singularity of the type $1/\sqrt{x}$ at both E_{\min} and E_{\max} . This singularity stems from a projection of the final ensemble at t_1 onto the curves of constant final energies E_1 of $H(q, p, t_1)$. Of course, all that we say here for the distribution of energies E_1 holds true also for the final action, the adiabatic invariant $I_1 = E_1/\omega_1$. It is perhaps worthwhile mentioning that the moments of our distribution according to (6) grow as $2^m/\sqrt{\pi m}$, whilst e.g. in the Gaussian distribution they grow much faster, namely as $2^m \Gamma(m+1/2)/\sqrt{\pi}$, where $\Gamma(x)$ denotes the gamma function.

Now we derive the distribution function (1) using the characteristic function $f(y)$ of $P(x)$, namely

$$f(y) = \int_{-\infty}^{\infty} e^{iyx} P(x) dx. \quad (7)$$

We immediately see that the n th derivative at $y = 0$ is equal to

$$f^{(n)}(0) = \int_{-\infty}^{\infty} (ix)^n P(x) dx = i^n \sigma_n, \quad (8)$$

where σ_n is the n th moment of $P(x)$ (in particular $\sigma_2 = \mu^2$), i.e.

$$\sigma_n = \int_{-\infty}^{\infty} x^n P(x) dx. \tag{9}$$

Using the Taylor expansion for $f(y) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} y^n$ and expressions (6), we see at once

$$f(y) = \sum_{m=0}^{\infty} \frac{i^{2m} \mu^{2m} (2m-1)!!}{m!(2m)!} y^{2m}, \tag{10}$$

and using the formula $(2m-1)!! = (2m)!/(2^m m!)$, we obtain

$$f(y) = \sum_{m=0}^{\infty} \left(-\frac{\mu^2 y^2}{2}\right)^m \frac{1}{(m!)^2}, \tag{11}$$

which can be summed and is equal to the Bessel function [8]

$$f(y) = J_0(\mu y \sqrt{2}). \tag{12}$$

Knowing the characteristic functions (7), (12), we now only have to invert the Fourier transform, namely

$$P(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iyx} f(y) dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iyx} J_0(\sqrt{2}\mu y) dy, \tag{13}$$

which is precisely equal to (1) for $|x| \leq \mu\sqrt{2}$ and is zero otherwise (see [8]). Indeed, distribution (1) is normalized to unity as it must be. It is essentially the so-called $\beta(1/2, 1/2)$ distribution or also termed arc sine density [9], after shifting the origin of x to $1/2$ and rescaling of x . It is obvious that the distribution function does not depend on any details of $\omega(t)$ and is in this sense universal for the 1D time-dependent harmonic oscillator in the case of *uniform canonical ensemble of initial conditions*, i.e. uniform w.r.t. the canonical angle.

The rigorous method of calculating \bar{E}_1 and μ^2 is explained in [1, 2]. In the general case when (2) cannot be solved exactly, the WKB method [10] can be applied to all orders, as has been explained in detail in [2].

Let us now present the direct algebraic derivation of the energy distribution function (1). By definition, we have

$$P(E_1) = \frac{1}{2\pi} \sum_{i=1}^4 \left| \frac{d\phi}{dE_1} \right|_{\phi=\phi_i(E_1)}, \tag{14}$$

where we have to sum up contributions from all four branches of the function $\phi(E_1)$. Let us denote $x = E_1 - \bar{E}_1$ so that we have

$$x = E_0 (\delta \cos(2\phi) + \gamma \sin(2\phi)) = \mu\sqrt{2} \sin(2\phi + \psi), \tag{15}$$

where δ and γ are expressed in terms of a, b, c, d as shown in [1, 2], the variance is $\mu^2 = \frac{E_0^2}{2} (\delta^2 + \gamma^2)$ and $\tan \psi = \delta/\gamma$ so that $\phi = \frac{1}{2} \arcsin \frac{x}{\mu\sqrt{2}} - \frac{\psi}{2}$. Therefore, $\left| \frac{d\phi_i}{dx} \right| = \frac{1}{2\sqrt{2\mu^2 - x^2}}$ for all four solutions $i = 1, 2, 3, 4$ and from (14) we get (1) at once.

The first derivation demonstrates the power of the approach employing the characteristic function $f(y)$ of $P(x)$, giving us new insights, whereas the second one leads elegantly to the final result, eliminating many parameters by a rather elementary substitution. In fact, this second method is geometrically obvious by the following argument: in a phase space, the initial distribution lies on an ellipse. The final distribution lies on a different ellipse with the same area, with the points equidistributed in the canonical angle variable. This final ellipse intersects the energy contours of the Hamiltonian, and thus after squeezing the final Hamiltonian so that

its energy contours are circles, the desired probability distribution is simply the distribution of radii of those circles through which the final ellipse passes.

In nonlinear systems, the entire theory expounded in [1, 2] must be reformulated and as such it is an important open problem [11]. For the case of a separatrix crossing some interesting numerical results have been obtained in [12], namely $P(E_1)$ has a substantial structure and is by far not so simple as (1).

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